RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT

4758/01

MATHEMATICS (MEI)

Differential Equations

THURSDAY 25 JANUARY 2007

Morning

Time: 1 hour 30 minutes

Additional materials:
- Answer booklet (8 pages)
- Graph paper
- MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by g m s^{-2}. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

1 The differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t} - 2y = \mathrm{e}^{-kt}$$

is to be solved for $t \geqslant 0$.

Consider the case $k = 2$.

(i) Find the general solution. [9]

(ii) Find the particular solution subject to the conditions $y = 0$ when $t = 0$, and y tends to zero as t tends to infinity. Show that this solution is zero only when $t = 0$ and sketch a graph of the solution. [7]

Consider now the case $k = 1$.

(iii) Find the general solution. Find also the particular solution subject to the same conditions as in part **(ii)**. [You may assume that $t\mathrm{e}^{-t} \to 0$ as $t \to \infty$.] [8]

2 The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\cot 2x = k, \qquad (*)$$

where k is a constant, is to be solved for $0 < x < \frac{1}{2}\pi$.

(i) Show that $\frac{\mathrm{d}}{\mathrm{d}x}(\ln \sin x) = \cot x$. [1]

(ii) In the case $k = 0$, solve the differential equation by separating the variables to find the general solution for y in terms of x. [6]

Now assume that $k > 0$.

(iii) Solve the differential equation to show that the general solution is

$$y = A\operatorname{cosec} 2x - \tfrac{1}{2}k\cot 2x$$

where A is an arbitrary constant. [7]

(iv) Find the particular solution subject to the condition $y = 0$ when $x = \frac{1}{4}\pi$. Sketch the graph of the solution for $0 < x < \frac{1}{2}\pi$, showing the behaviour of y as x tends to zero. [4]

(v) By using the double angle formulae for $\sin 2x$ and $\cos 2x$, or otherwise, show that there is a solution to (*) for which y tends to a finite limit as x tends to zero. State the solution and its limiting value. [6]

3 In an experiment, a ball-bearing of mass m kg falls through a liquid. The ball-bearing is released from rest and t seconds later its displacement is x m and its velocity is v m s^{-1}. The forces acting on the ball-bearing are its weight and a resistance force R N. Three models for R are to be considered.

In the first model, $R = mk_1 v$, where k_1 is a positive constant.

(i) Show that $\frac{\mathrm{d}v}{\mathrm{d}t} = g - k_1 v$. Hence show that $v = \frac{g}{k_1}(1 - \mathrm{e}^{-k_1 t})$. [7]

(ii) Find an expression for x in terms of t. [4]

In the second model, $R = mk_2 v^2$, where k_2 is a positive constant.

(iii) Show that $\frac{v}{g - k_2 v^2}\frac{\mathrm{d}v}{\mathrm{d}x} = 1$ and hence find v in terms of x. [7]

In the third model, $R = mk_3 v^{\frac{3}{2}}$, where k_3 is a positive constant. Euler's method is used to solve the resulting differential equation $\frac{\mathrm{d}v}{\mathrm{d}t} = g - k_3 v^{\frac{3}{2}}$.

The algorithm is given by $t_{r+1} = t_r + h$, $v_{r+1} = v_r + h\dot{v}_r$.

(iv) Given $k_3 = 1.225$ and using a step length of 0.1, perform two iterations of the algorithm to estimate v when $t = 0.2$. [5]

The terminal velocity of the ball-bearing is 4 m s^{-1}.

(v) Verify the value of k_3 given in part **(iv)**. [1]

[Question 4 is printed overleaf.]

4 The following simultaneous differential equations are to be solved.

$$\frac{dx}{dt} = -3x - y + 10,$$
$$\frac{dy}{dt} = \ \ 2x - y + 5.$$

(i) Find the values of x and y for which $\frac{dx}{dt} = \frac{dy}{dt} = 0$. [3]

(ii) Show that $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 5$. [5]

(iii) Find the general solution for x in terms of t. Hence obtain the corresponding general solution for y. [10]

(iv) Given that $x = y = 0$ when $t = 0$, find the particular solutions. [3]

(v) Sketch the graph of the solution for x, making clear the behaviour of the curve initially and for large values of t. [3]

Mark Scheme 4758
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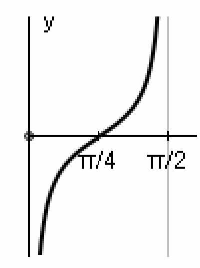
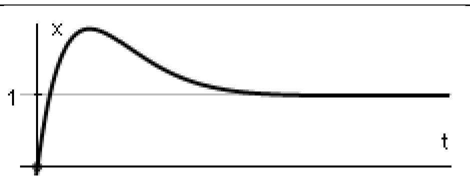
1(i)	$\lambda^2-\lambda-2=0$ $\lambda=-1$ or 2 CF $y=A\mathrm{e}^{-t}+B\mathrm{e}^{2t}$ PI $y=a\mathrm{e}^{-2t}$ $\dot{y}=-2a\mathrm{e}^{-2t}, \ddot{y}=4a\mathrm{e}^{-2t}$ $4a\mathrm{e}^{-2t}-\left(-2a\mathrm{e}^{-2t}\right)-2a\mathrm{e}^{-2t}=\mathrm{e}^{-2t}$ $4a=1$ $a=\frac{1}{4}$ $y=A\mathrm{e}^{-t}+B\mathrm{e}^{2t}+\frac{1}{4}\mathrm{e}^{-2t}$	M1 A1 F1 B1 M1 M1 M1 A1 F1	Auxiliary equation CF for their roots Differentiate twice Substitute Compare and solve Their CF with 2 constants + their PI	9
(ii)	$0=A+B+\frac{1}{4}$ $t\to\infty\Rightarrow \mathrm{e}^{-t}\to 0, \mathrm{e}^{-2t}\to 0, \mathrm{e}^{2t}\to\infty$ so $y\to 0\Rightarrow B=0$ $y=\frac{1}{4}\left(\mathrm{e}^{-2t}-\mathrm{e}^{-t}\right)$ $y=0\Leftrightarrow \mathrm{e}^{-t}=\mathrm{e}^{-2t}\Leftrightarrow \mathrm{e}^{t}=1\Leftrightarrow t=0$ 	M1 M1 A1 M1 E1 B1 B1	Use initial condition Use asymptotic condition cao Valid method to establish 0 is *only* root Complete argument Curve satisfies both conditions $y\neq 0$ for $t>0$ and consistent with their solution	7
(iii)	CF $y=C\mathrm{e}^{-t}+D\mathrm{e}^{2t}$ PI $y=bt\mathrm{e}^{-t}$ $\dot{y}=b\mathrm{e}^{-t}-bt\mathrm{e}^{-t}, \ddot{y}=-2b\mathrm{e}^{-t}+bt\mathrm{e}^{-t}$ $-2b\mathrm{e}^{-t}+bt\mathrm{e}^{-t}-\left(b\mathrm{e}^{-t}-bt\mathrm{e}^{-t}\right)-2b\mathrm{e}^{-t}=\mathrm{e}^{-t}$ $\Rightarrow -2b-b=1\Rightarrow b=-\frac{1}{3}$ GS $y=C\mathrm{e}^{-t}+D\mathrm{e}^{2t}-\frac{1}{3}t\mathrm{e}^{-t}$ $y=0, t=0\Rightarrow C+D=0$ $y\to 0\Rightarrow D=0$ $y=-\frac{1}{3}t\mathrm{e}^{-t}$	F1 B1 M1 A1 F1 M1 M1 A1	Correct or same as in (i) Differentiate (product) and substitute cao Their CF + their non-zero PI Use condition Use condition cao	8

2(i)	$\frac{\mathrm{d}}{\mathrm{d}x}(\ln \sin x) = \frac{1}{\sin x}\cos x = \cot x$	E1	Differentiate (chain rule)	1
(ii)	$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = -2\cot 2x$ $\int \frac{1}{y}\,\mathrm{d}y = \int -2\cot 2x\,\mathrm{d}x$ $\ln\lvert y\rvert = -\ln\lvert\sin 2x\rvert + c$ $y = A\operatorname{cosec} 2x$	M1 M1 A1 A1 M1 A1	Rearrange Integrate One side correct (ignore constant) All correct, including constant Rearrange, dealing properly with constant	6
(iii)	$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\cot 2x = k$ $I = \exp\left(\int 2\cot 2x\,\mathrm{d}x\right)$ $= \exp(\ln \sin 2x)$ $= \sin 2x$ $\frac{\mathrm{d}y}{\mathrm{d}x}\sin 2x + 2y\cos 2x = k\sin 2x$ $y\sin 2x = \int k\sin 2x\,\mathrm{d}x$ $= -\frac{1}{2}k\cos 2x + A$ $y = A\operatorname{cosec} 2x - \frac{1}{2}k\cot 2x$	M1 M1 A1 M1 M1 A1 E1	Attempt integrating factor Integrate Simplified form of IF Multiply by their IF Integrate both sides cao	7
(iv)	$x = \frac{1}{4}\pi, y = 0 \Rightarrow 0 = A$ $y = -\frac{1}{2}k\cot 2x$ 	M1 A1 B1 B1	Use condition Increasing and through $\left(\frac{1}{4}\pi, 0\right)$ Asymptote $x = 0$	4
(v)	$y = \frac{A - \frac{1}{2}k\cos 2x}{\sin 2x} = \frac{A - \frac{1}{2}k\left(1 - 2\sin^2 x\right)}{2\sin x\cos x}$ $A = \frac{1}{2}k \Rightarrow y = \frac{\frac{1}{2}k\sin x}{\cos x}$ which tends to zero as $x \to 0$	B1 M1 A1 M1 E1 B1	Both double angle formulae correct (or small angle approximations or series expansion) Use expressions in general solution Identify value of *A* Correct solution, fully justified Must be from correct solution	6

3(i)	$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - R$	B1	N2L equation (accept *ma*, allow sign errors)	
	$\frac{\mathrm{d}v}{\mathrm{d}t} = g - k_1 v$	E1	Must follow from correct N2L	
	$\int \frac{1}{g - k_1 v}\,\mathrm{d}v = \int \mathrm{d}t$	M1	Separate and integrate	
	$-\frac{1}{k_1}\ln\lvert g - k_1 v\rvert = t + c_1$	A1	LHS	
	$g - k_1 v = A\mathrm{e}^{-k_1 t}$	M1	Rearrange (dealing properly with constant)	
	Alternatively	*M1*	Attempt integrating factor	
		A1	$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{k_1 t} v\right) = g\,\mathrm{e}^{k_1 t}$	
		M1	Integrate	
	Alternatively	*M1*	Auxiliary equation	
		A1	CF $A\mathrm{e}^{-k_1 t}$	
		M1	Constant PI (g / k_1)	
	$t = 0, v = 0 \Rightarrow A = g$	M1	Use condition	
	$v = \frac{g}{k_1}\left(1 - \mathrm{e}^{-k_1 t}\right)$	E1		7
(ii)	$x = \int v\,\mathrm{d}t = \frac{g}{k_1}\left(t + \frac{1}{k_1}\mathrm{e}^{-k_1 t} + B\right)$	M1	Integrate *v*	
		A1	cao (including constant)	
	$t = 0, x = 0 \Rightarrow B = -\frac{1}{k_1}$	M1	Use condition	
	$x = \frac{g}{k_1}\left(t + \frac{1}{k_1}\mathrm{e}^{-k_1 t} - \frac{1}{k_1}\right)$	A1	cao	4
(iii)	$mv\frac{\mathrm{d}v}{\mathrm{d}x} = mg - mk_2 v^2$	B1	N2L with $mk_2 v^2$ (accept ma or $m\frac{\mathrm{d}v}{\mathrm{d}t}$)	
	$\frac{v}{g - k_2 v^2}\frac{\mathrm{d}v}{\mathrm{d}x} = 1$	E1	Must follow from correct N2L	
	$\int \frac{v}{g - k_2 v^2}\,\mathrm{d}v = \int \mathrm{d}x$	M1	Integrate	
	$-\frac{1}{2k_2}\ln\left\lvert g - k_2 v^2\right\rvert = x + c_2$	A1	LHS	
	$g - k_2 v^2 = C\mathrm{e}^{-2k_2 x}$	M1	Rearrange (dealing properly with constant)	
	$x = 0, v = 0 \Rightarrow C = g$	M1	Use condition	
	$v = \sqrt{\frac{g}{k_2}\left(1 - \mathrm{e}^{-2k_2 x}\right)}$	A1	cao	7
(iv)	t v $\dot{v}$	B1	First line	
	0 0 9.8	M1	Use algorithm	
	0.1 0.98 8.61157	A1	0.98	
	0.2 1.84116	M1	Use algorithm	
		A1	1.84116 (accept 3sf or better)	5

(v)	$g - k_3 v^{\frac{3}{2}} = 0$ when $v = 4 \Rightarrow k_3 = \dfrac{g}{4^{\frac{3}{2}}} = 1.225$	E1	Deduce or verify value (must relate to resultant force or acceleration being zero)	1

4(i)	subtracting $\Rightarrow -5x+5=0$ $x=1$ $y=7$	M1 A1 A1	Solve simultaneously	3
(ii)	$\ddot{x}=-3\dot{x}-\dot{y}$ $=-3\dot{x}-(2x-y+5)$ $=-3\dot{x}-2x+\left(-\dot{x}-3x+10\right)-5$ $\ddot{x}+4\dot{x}+5x=5$	M1 M1 M1 M1 E1	Differentiate Substitute for $\dot{y}$ y in terms of $x,\dot{x}$ Substitute	5
(iii)	$\lambda^2+4\lambda+5=0$ $\lambda=-2\pm j$ CF $x=e^{-2t}\left(A\cos t+B\sin t\right)$ PI $x=\frac{5}{5}=1$ GS $x=e^{-2t}\left(A\cos t+B\sin t\right)+1$ $y=-\dot{x}-3x+10$ $=-e^{-2t}\left(-A\sin t+B\cos t\right)+2e^{-2t}\left(A\cos t+B\sin t\right)$ $-3e^{-2t}\left(A\cos t+B\sin t\right)-3+10$ $=e^{-2t}\left(\left(-A-B\right)\cos t+\left(A-B\right)\sin t\right)+7$	M1 M1 A1 F1 B1 F1 M1 M1 M1 A1	Auxiliary equation Solve to get complex roots CF for their roots Their CF with 2 constants + their PI y in terms of $x,\dot{x}$ Differentiate their x Substitute cao	10
(iv)	$t=0, x=0 \Rightarrow A+1=0$ $t=0, y=0 \Rightarrow -A+B+7=0$ $A=-1, B=8$ $x=e^{-2t}\left(8\sin t-\cos t\right)+1$ $y=-e^{-2t}\left(7\cos t+9\sin t\right)+7$	M1 M1 A1	Use condition on x Use condition on y Both correct	3
(v)	 NB Oscillates about $x=1$, but not apparent at this scale due to small amplitude	B1 B1 B1	Through origin Positive gradient at t = 0 Asymptote x = 1, or their non-zero constant PI (accept oscillatory or non-oscillatory)	3

4758 - Differential Equations

General Comments

The standard of work was generally very good, demonstrating a clear understanding of the techniques required. Almost all candidates answered questions 1 and 4; questions 2 and 3 were equally popular as a third choice. Candidates often produced accurate work when solving second order differential equations. However, when solving first order equations, it was common to see errors in integration, omission of the constant of integration, or not dealing properly with the constant.

With regard to graph sketching, I would like to emphasise the following advice given in last summer's report. Candidates should note that generally the expectation of sketches in this unit is that any known information (such as initial conditions or relevant results found earlier in the question) should be indicated on the sketch. Also any obvious shape and features of the graph (e.g. oscillating, increasing, decreasing, unbounded, asymptotes), should be shown. Detailed calculations are not required, unless specifically requested.

Comments on Individual Questions

1 **Second order differential equations**

(i) This was often completely correct.

(ii) Many candidates correctly found the particular solution, but a few candidates were unable to use the asymptotic condition. Only a minority of candidates were able to demonstrate that the solution is zero *only* when $t = 0$. Commonly candidates simply showed that $t = 0$ implies $y = 0$, or gave a vague argument why it is the only root. Sketch graphs were often good, except they often showed y positive rather than negative.

(iii) The solution was often correct, but some did not realise the correct form of the particular integral and some made slips with their arbitrary constants.

2 **First order differential equations**

(i) This derivative (using the chain rule) was intended to help with the integral later in the question, but unfortunately many candidates misunderstood the request and approached it as an integral or a differential equation.

(ii) This was often done well, but slips in integration were common, in particular not dealing with the $2x$ properly and omitting the constant. Another common error was not dealing with the constant properly when rearranging.

(iii) Most candidates attempted to use an integrating factor, but slips were common. A common error was $I = \exp\left(\int \cot 2x \, \mathrm{d}x\right) = \exp\left(\frac{1}{2}\ln \sin 2x\right) = \frac{1}{2}\sin 2x$, which gave a valid integrating factor but contained two errors, i.e. omitting the 2 in the integral and then wrongly dealing with the $\frac{1}{2}$.

(iv) Many candidates successfully found the particular solution, but some made slips in the calculation. The sketch graphs varied widely in quality. Some candidates simply drew a sketch of cot*x*. Many sketches did not show the given condition, indeed some wrongly had a break in the curve around the point $\left(\frac{1}{4}\pi, 0\right)$, presumably due to entering *y* = 1/tan2*x* into a graphical calculator. If using a graphical calculator, candidates must use it intelligently and not just copy the screen without thinking about what the graph should look like.

(v) Very few candidates successfully completed this part. Many tried to solve the differential equation again, which was rarely successful, rather than use their general solution.

3 **Modelling the motion of a ball-bearing falling through a liquid**

(i) Candidates usually formulated the differential equation correctly from Newton's second law, and many solved it correctly. However, errors did occur, in particular with the integral. Also, some candidates omitted the constant or made mistakes with the constant when rearranging their solution.

(ii) Most candidates integrated the previous solution, but many omitted the constant or did not calculate it.

(iii) This part was often done well, but candidates were not as successful as in part (i). Common errors with formulating the equation were: not starting from Newton's second law or not recalling the alternative form of acceleration. Common errors when solving the differential equation were: not integrating correctly, omitting the constant, or including the constant but wrongly stating that it was the initial displacement and hence zero.

(iv) This was usually correct, although as usual with an Euler's method calculation, a few candidates produced unrecognisable figures with no method shown. Some candidates mistakenly gave the value of *v* at *t* = 0.3 by tabulating $t, v_r, \dot{v}_r$ and v_{r+1} in each row and then stating the final value of *v* in their final row.

(v) Virtually all attempts at this calculation were correct.

4 **Simultaneous differential equations**

(i) This calculation was usually correct, although some made slips.

(ii) The elimination of *y* was often done well, although a few differentiated the first equation with respect to *x* rather than *t*.

(iii) The solution for *x* was often done very well. When finding *y*, many candidates correctly used the first equation. As usually occurs, a few tried instead to set up and solve a differential equation for *y*. Such attempts were very lengthy and time-consuming and never attempted to relate the arbitrary constants in the two solutions, which is a vital feature of the solutions.

(iv) The particular solutions were often done well.

(v) The sketch was often correct, but some candidates seemed to ignore the request to make clear the initial and the long-term behaviour of the solution on the sketch.